Using the inverse matrix to solve equations

Introduction

One of the most important applications of matrices is to the solution of linear simultaneous equations. On this leaflet we explain how this can be done.

1. Writing simultaneous equations in matrix form

Consider the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Provided you understand how matrices are multiplied together you will realise that these can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

we have

$$AX = B$$

This is the **matrix form** of the simultaneous equations. Here the unknown is the matrix X, since A and B are already known. A is called the **matrix of coefficients**.

2. Solving the simultaneous equations

Given

$$AX = B$$

we can multiply both sides by the inverse of A, provided this exists, to give

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, the identity matrix. Furthermore, $IX = X$, because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered. So

$$X = A^{-1}B$$

$$\text{if} \quad AX = B, \qquad \text{then} \qquad X = A^{-1}B$$

This result gives us a method for solving simultaneous equations. All we need do is write them in matrix form, calculate the inverse of the matrix of coefficients, and finally perform a matrix multiplication.

Example
Solve the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Solution

We have already seen these equations in matrix form:

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$.

$$\begin{aligned} A^{-1} &= \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \end{aligned}$$

Then X is given by

$$\begin{aligned} X = A^{-1}B &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -22 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Hence $x = 2$, $y = 1$ is the solution of the simultaneous equations.

Exercises
1. Solve the following sets of simultaneous equations using the inverse matrix method.

a) $\begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned}$ b) $\begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned}$

Answers

1. a) $x = 3, y = -2$, b) $x = -2, y = 2$.

